Math 142 Lecture 3 Notes

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1 Homeomorphisms, Disjoint Unions, and Product Spaces

1.1 Homeomorphisms

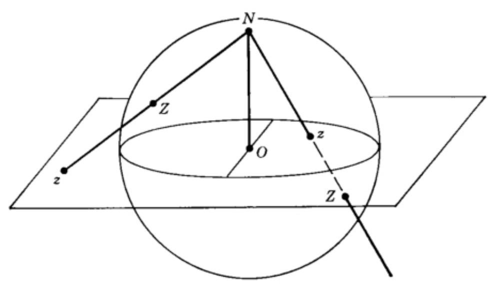
How can we say that two topological spaces are "the same"?

Definition 1.1. A function $f : X \to Y$ is a *homeomorphism* if f is a continuous bijection with a continuous inverse. We call X, Y *homeomorphic spaces*, denoted by $X \cong Y$.

If f is a homeomorphism with inverse f^{-1}, then if $A \subseteq X$ is open, then $(f^{-1})^{-1}(A) \subseteq Y$ is open (as f^{-1} is continuous). Since $f = (f^{-1})^{-1}$, this means that a homeomorphism is a bijection between the open sets in X and the open sets in Y.

Example 1.1. A continuous bijection might not have a continuous inverse. Let $X = \mathbb{R}$ with the discrete topology and $Y = \mathbb{R}$ with the trivial topology. Let $f : X \to Y$ be defined as $f(x) = x$. f is continuous, as $f^{-1}(\varnothing) = \varnothing$ is open, and $f^{-1}(\mathbb{R}) = \mathbb{R}$ is open. But $f^{-1} : Y \to X$ takes $f^{-1}(x) = x$, and $(f^{-1})^{-1}(\{1\}) = \{1\}$ is not open in Y.

Example 1.2 (stereographic projection)**.** Define the set $S^{(n)} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$ be the n-dimensional sphere.[1] Consider $f : S^n \setminus \{(0, 0, \dots, 0, 1)\} \to \mathbb{R}^n$ (the domain missing the "north pole") given as follows. Take x on the sphere and draw a line containing x and the north pole; this line intersects the plane, and we set $f(x)$ to be this point of intersection.[2] Check that this is a bijection.



[1] The n-dimensional sphere sits in $n + 1$ dimensional space.
[2] I did not create this picture; I found it on Google.

We want to show that f is continuous. Let $U \subseteq \mathbb{R}^n$ be open. Let $U' \subseteq \mathbb{R}^{n+1}$ be all the half-lines from p to a point $x \in U$ (not including p). Check that U' is open in $\mathbb{R}^{n+1}$. S_n has the subspace topology, so $U' \cap S^n$ is open in S^n. But $f^{-1}(U) = U' \cap S^n$. So $f^{-1}(U)$ is open, making f continuous. A similar argument using U' shows that f^{-1} is continuous. So f is a homeomorphism.

1.2 Creating new topological spaces

Using the idea of the subspace topology, we can create new topological spaces form larger ones. How else can we construct topological spaces?

1.2.1 Disjoint unions of spaces

Definition 1.2. If X, Y are topological spaces, then the *disjoint union* $X \amalg Y$ (also called $X + Y$) is the set $X \amalg Y$ with open sets U_α and V_β, where $U_\alpha \subseteq X$ is open, $V_\beta \subseteq Y$ is open, and unions of these sets are open.

Example 1.3. Let $X = \{1, 2, 3\}$ with open sets $\varnothing, X$, and let $Y = \{3, 4, 5\}$ with open sets $\varnothing, Y, \{3, 4\}$. Then

$$X \amalg Y = \{1, 2, 3_x, 3_y, 4, 5\},$$

with open sets $\varnothing, \{1, 2, 3_x\}, \{3_y, 4, 5\}, \{3_y, 4\}, \{1, 2, 3_x, 3_y, 4\}, \{1, 2, 3_x, 3_y, 4, 5\}$.

1.2.2 Products of spaces

Definition 1.3. If X and Y are topological spaces, then the *product space* $X \times Y$ is the set

$$X \times Y = \{(x, y) \in X \times Y : x \in X, y \in Y\}$$

with a base for the topology given by $\{U \times V : U \subseteq X \text{ open}, V \subseteq Y \text{ open}\}$.

Example 1.4. $\mathbb{R}^2 \cong \mathbb{R} \times \mathbb{R}$. Here, the set $(0, 1) \times (0, 1)$ is open and in the base. The open unit ball is an open set, but it is not in the base; it is a union of infinitely many squares in the base.

Product spaces come with projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$, where $p_1(x, y) = x$, and $p_2(x, y) = y$.

Theorem 1.1. *If $X \times Y$ has the product topology, then p_1 and p_2 are continuous, and take open sets to open sets. Furthermore, the product topology is the smallest topology for which p_1 and p_2 are continuous.*

Proof. If $U \subseteq X$ is open, then $p_1^{-1}(U) = U \times Y$. But U is open in X and Y is open in Y, so $U \times Y$ is open in $X \times Y$. So p_1 is continuous. Similarly, p_2 is continuous.

If $A \subseteq X \times Y$ is open, then $A = \bigcup(U_i \times V_i)$ for some open sets $U_i \subseteq X$ and $V_i \subseteq Y$. Then

$$p_1(A) = \bigcup p_1(U_i \times V_i) = \bigcup U_i,$$

which is a union of open sets, making it open in X. The same argument works for p_2.

Now assume $X \times Y$ has another topology where p_1, p_2 are continuous. Then if $U \subseteq X$ and $V \subseteq Y$ are open, then $p_1^{-1}(U) = U \times Y$ and $p_2^{-1}(V) = X \times V$ are open in this topology. So $(U \times Y) \cap (X \times V) = U \times V$ is open, and then any union $\bigcup(U_i \times V_i)$ is open in this topology. So any open set in the product topology is open in this new topology. $\square$

Theorem 1.2. *A function $f : Z \to X \times Y$ is continuous iff $p_1 \circ f$ and $p_2 \circ f$ are continuous.*

Proof. ($\Longrightarrow$) If f is continuous, then $p_1 \circ f$ and $p_2 \circ f$ are compositions of continuous functions and are therefore continuous.

($\Longleftarrow$) If $p_1 \circ f$ and $p_2 \circ f$ are continuous, we need to show that $f^{-1}(U \times V) \subseteq Z$ is open for any open $U \subseteq X$, $V \subseteq Y$. But

$$\begin{aligned} f^{-1}(U \times V) &= f^{-1}(p_1^{-1}(U) \cap p_2^{-1}(V)) \\ &= f^{-1}(p_1^{-1}(U)) \cap f^{-1}(p_2^{-1}(V)) \\ &= (p_1 \circ f)^{-1}(U) \cap (p_2 \circ f)^{-1}(V), \end{aligned}$$

which is an intersection of open sets since $p_1 \circ f$ and $p_2 \circ f$ are continuous. So it is open. $\square$